OPERATORS

Name ________________

Operators are mechanisms for making new vectors out of old ones. Examples of operators could include mathematical objects that double the magnitude of a vector, rotate a vector by 45°, swap the components of a vector, project a vector onto an axis, or rotate a vector continually in time.

Some operators are *linear,* meaning that $A(|\psi_1\rangle + |\psi_2\rangle) = A|\psi_1\rangle + A|\psi_2\rangle$ and $A(c|\psi\rangle) = cA|\psi\rangle$ (where A is an operator, $|\psi\rangle$ is a quantum state, and c is a scalar). Linear operators can be represented by arrays of numbers, called "matrices," which obey the rules of matrix multiplication.

I. Projection operators

We already have a mathematical expression for "the projection of $|\psi\rangle$ onto $|\psi_1\rangle$": it is

$|\psi\rangle\langle\psi|\psi_1\rangle$

Another way to say this is that there is a projection *operator* acting on $|\psi_1\rangle$ whose form is:

$|\psi\rangle\langle\psi|$

A. Check that the projection operator $|+\rangle\langle+|$ is represented in matrix form as $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
(*Hint:* Use matrix notation to check that for a general state $|\psi\rangle = a\,|+\rangle + b\,|-\rangle$, the given matrix acts on $|\psi\rangle$ the way a projection operator should.)

B. Write the projection operator $|-\rangle\langle-|$ in matrix form. (*Hint:* Guess and check!)

C. Calculate the sum of the above two projection operators. Explain why this result makes sense.

A projection operator corresponds to an interaction that selects a definite state (for some property of a system) from a general state. This kind of process is called a *measurement.* As we have seen, the only possible result of a measurement of a system property is a definite state for that property.

The action of polarizers and splitters is represented by projection operators, and these all have the same mathematical form in matrix notation.

II. Spin operators

A spin operator corresponds to an interaction that characterizes the spin along some axis. There is a spin operator for any direction, such as S_x, S_y, or S_z. Applying the spin operator to a definite spin state (such as $|+\rangle$ or $|-\rangle$) preserves the definite state and returns a measured value that is a multiple of $\pm\hbar/2$: mathematically,

$$S_z\,|+\rangle = +\hbar/2\ |+\rangle$$

$$S_z\,|-\rangle = -\hbar/2\ |-\rangle$$

Let's use these relations to find the matrix representation of S_z.

A. Write S_z as a general unknown matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and write $|+\rangle$ and $|-\rangle$ in matrix notation; then use the above relations to find *a, b, c,* and *d.*

Next let's use the S_x relations to find the matrix representation of S_x:

$$S_x\,|+\rangle_x = +\hbar/2\ |+\rangle_x = +\hbar/2\ (1/\sqrt{2})(\,|+\rangle + |-\rangle)$$

$$S_x\,|-\rangle_x = -\hbar/2\ |-\rangle_x = -\hbar/2\ (1/\sqrt{2})(\,|+\rangle - |-\rangle)$$

B. Write S_x as a general unknown matrix, $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, and use the above relations to find *p, q, r,* and *s.*

III. Mean values

It can be informative to predict the mean value of the outcome of many experiments. For example, if we measure the spin of many electrons in the same state, we might like to know the average of these measurements. We can calculate this by weighting each of the measured values with its probability:

$$\langle A \rangle = \sum a_n \mathcal{P}_n$$

where A is the operator representing what is measured, a_n is the measured value associated with the nth definite state for that system, and $\mathcal{P}_n$ is the probability of the nth definite state for that system.

A. Let's use this reslationship to calculate $\langle S_z \rangle$ for a few different experiments.

1. What are the a_n in this case (all the possible values of spin in the z-direction)?

2. Suppose we send in a stream of millions of electrons that we have prepared to all be in the state $|+\rangle$. What is the probability of each of the possible values of spin in the z-direction?

3. Use your answers so far to calculate the average spin in the z-direction $\langle S_z \rangle$ for this situation. Why does your answer make sense?

B. The equation above is mathematically equivalent to the following (as demonstrated in McIntyre, eq. 2.75):

$$\langle A \rangle = \langle \psi | A | \psi \rangle$$

Write the equation for $\langle S_z \rangle = \langle + | S_z | + \rangle$ in matrix form. Calculate and check against A.3.

C. Suppose an incoming stream of electrons is prepared to all be in the state $|+\rangle_x$, rather than $|+\rangle$. Calculate the average z-spin $\langle S_z \rangle$ for this experiment. (Use either method–or use both and check that they agree.) Why does your answer make sense?

D. What is the average spin in the z-direction of the state $\frac{1}{\sqrt{13}}\binom{3i}{2}$? Show your work.

E. Which of the following states will have the smallest magnitude of average spin in the z-direction? Explain your reasoning.

$$|\psi_1\rangle = \frac{1}{\sqrt{401}}\binom{20}{1} \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}\binom{1}{1}$$